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GRAPHS WITH DEGREES FROM PRESCRIBED INTERVALS

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January 1975

GRAPHS WITH DEGREES FROM PRESCRIBED INTERVALS*

by

Michael Koren

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Abstract

Necessary and sufficient conditions for the existence of simple graphs with degrees from prescribed intervals, are given.

1. Introduction

All graphs in this paper are finite and have no loops and no multiple edges. For undefined terms see [4].

The degree, $d(p) = d(p,G)$ of a vertex p in an undirected graph G, is the number of edges of G, incident with p. The outdegree $d^+(p,D)$ (indegree $d^-(p,D)$) of a vertex p in a directed graph D, is the number of edges of D, having p as an initial (terminal) vertex.

Using flows, L. R. Ford and D. R. Fulkerson [2, Theorem 11.1] give necessary and sufficient conditions under which a directed graph D has a subgraph whose outdegrees and indegrees lie in prescribed intervals. The aim of this paper is to study analogue conditions for undirected graphs.

2. Notation

A graph is considered to be undirected unless otherwise specified. All graphs in this paper have the same set of vertices $\{p_1,\dots,p_n\}$. A graph G is identified with its set of edges; for example, the complete graph on n vertices is $K_n = \{(p_i,p_j) \mid 1 \leq i < j \leq n\}$.

Definition: A semi-graph W is a function from the edges of K_n into $\{0,\frac{1}{2},1\}$. The degree of a vertex p in a semi-graph W is $d(p,W) = \sum_{i=1}^{n} W(p,p_i)$. A semi-graph W is a semi-subgraph of a graph G if $(p_i,p_j) \notin G \Rightarrow W(p_i,p_j) = 0$.

Notation: Throughout, ϕ and ψ will denote two sequences $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$, respectively, of non-negative integers, such that $a_i \leq b_i$ for $i = 1,\dots,n$.

Definition: A graph H (semi-graph H) is a $[\phi,\psi]$-realization (semi-$[\phi,\psi]$-realization) if $a_i \leq d(p_i,H) \leq b_i$ for $i = 1,\dots,n$. A $[\phi,\psi]$-factor (semi-$[\phi,\psi]$-factor), of a given graph G, is a subgraph (semi-subgraph) of G which is a $[\phi,\psi]$-realization (semi-$[\phi,\psi]$-realization). The prefix $[\phi,\psi]$- will sometimes be omitted.

Definition: For a directed graph D, and a set $S \subset \{1,\ldots,n\}$, $\delta^+(p,S)$ is the number of edges of D, going from p to a vertex in $S^* = \{p_i | i \in S\}$, and $\delta^-(p,S)$ is the number of edges of D, going from a vertex of S^* to p. Similarly, for a graph G, $\delta(p,S)$ is the number of edges of G, connecting p to vertices in S^*.

3. Weighted subgraphs

We will need the following known theorem:

Theorem 3.1: [2, Theorem 11.1]. Suppose D is a directed graph on n vertices $v_1,\ldots,v_n$, and numbers a_i, b_i, a_i', b_i' are given $(i = 1,\ldots,n)$, such that $a_i \le b_i$, $a_i' \le b_i'$ for $i = 1,\ldots,n$. Then D has a subgraph E for which

$$a_i \le d^+(v_i,E) \le b_i \quad (i = 1,\ldots,n) \tag{3.1}$$

and

$$a_i' \le d^-(v_i,E) \le b_i' \quad (i = 1,\ldots,n) \tag{3.2}$$

if and only if for all $S \subseteq \{1,\ldots,n\}$

$$\sum_{i\in S} a_i \le \sum_{j=1}^{n} \min[b_j', \delta^+(v_j,S)] \tag{3.3}$$

and

$$\sum_{i\in S} a_i' \leq \sum_{j=1}^{n} \min[b_j, \delta^-(v_j, S)]. \tag{3.4}$$

Lemma 3.1: If $S,T \subset \{1,\ldots,n\}$, and $S \cap T = \emptyset$, then for any graph G

$$\sum_{i\in S} d(p_i,G) \leq \sum_{i\in T} d(p_i,G) + s(n-1-t),$$

where s and t are the cardinalities of S and T, respectively.

Proof: Because G is simple,

$$\sum_{i\in S} d(p_i,G) - s(s-1) \leq \mathrm{Card}\{(p_i,p_j) \in G \mid i \in S,\ j \notin S\}$$

$$\leq \sum_{i\notin S} \min(s, d(p_i,G)) \leq \sum_{i\in T} d(p_i,G) + s(n-s-t).$$

Lemma 3.2: (Compare [5, Lemma 2.2]). Let W be a semi-graph, and let $S \neq \emptyset$ and T be two disjoint subsets of $N = \{1,\ldots,n\}$. Then, if

$$i \in S,\ j \in N-T \Rightarrow W(p_i,p_j) = 1 \tag{3.5}$$

and

$$i \in T,\ j \in N-S \Rightarrow W(p_i,p_j) = 0, \tag{3.6}$$

then

$$\sum_{i\in S} d(p_i,W) = \sum_{i\in T} d(p_i,W) + s(n-1-t). \tag{3.7}$$

Proof: By condition (3.5), $\sum_{i\varepsilon S} d(p_i,W) - s(s-1) = \sum_{\substack{i\varepsilon S\\ j\varepsilon N-S}} W(p_i,p_j)$, and by condition (3.6), the last sum is equal to

$$\sum_{\substack{i\varepsilon S\\ j\varepsilon T}} W(p_i,p_j) + s(n-s-t) = \sum_{i\varepsilon T} d(p_i,W) + s(n-s-t).$$

Definition: Let $C = [p_{i_1},\dots,p_{i_\ell}]$ be a path or a cycle in a semi-graph W, such that $W = \frac{1}{2}$ on all its edges. Alternating C will mean the changing of W on C, by alternatively adding and subtracting $\frac{1}{2}$ along C. In a positive (negative) alternation we begin at p_{i_1}, (or at another specified vertex), by adding (subtracting) $\frac{1}{2}$.

Remark: If C is a path, then $d(p_{i_1},W)$ and $d(p_{i_\ell},W)$ will be changed by $\frac{1}{2}$, by an alternation. If C is an odd cycle, i.e. cycle with odd number of edges, then $d(p_{i_1},W)$ will be increased, or decreased by 1, depending on whether the alternation is positive or negative. In any alternation, W becomes integral on C, and the degrees of $p_{i_2},\dots,p_{i_{\ell-1}}$ do not change. If C is even, i.e. C has even number of edges, then the degree of p_{i_1} also does not change.

Lemma 3.3: A graph C has a semi-$[\phi,\psi]$-factor W such that for $i = 1,\dots,n$

$$d(p_i,W) \text{ is integer} \tag{3.8}$$

If and only if for all $S \subseteq N$

$$\sum_{i\varepsilon S} a_i \le \sum_{j=1}^{n} \min(b_j,\delta(p_j,S)). \tag{3.9}$$

Proof: Let D be the symmetric directed graph which is obtained from G by replacing each edge (p_i,p_j) by two directed edges (one from p_i to p_j and one from p_j to p_i). Since $\delta(p_i,S) = \delta^+(p_i,S) = \delta^-(p_i,S)$ for $i = 1,\ldots,n$, and for all $S \subseteq N$, it follows from Theorem 3.1, that D has a directed subgraph E for which

$$a_i \le d^+(p_i,E), d^-(p_i,E) \le b_i, \quad (i = 1,\ldots,n), \tag{3.10}$$

if and only if condition (3.9) holds for ϕ,ψ and G.

Suppose first that G has a semi-$[\phi,\psi]$-factor W which fulfills condition (3.8). Let $W_1 = \{(p_i,p_j)|W(p_i,p_j) = 1\}$ and $W_2 = \{(p_i,p_j)|W(p_i,p_j) = \frac{1}{2}\}$. By condition (3.8), each component of W_2 is Eulerian. By orienting each component of W_2 along an Eulerian cycle, and by replacing each edge of W_1 by two directed edges (one in each direction) we obtain a directed subgraph E of D which fulfills condition (3.10). Hence condition (3.9) holds for ϕ,ψ and G.

Suppose now that D has a subgraph E, for which condition (3.10) holds. Define

$$W(p_i,p_j) = \begin{cases} 1 & \text{if } (p_i,p_j),(p_j,p_i) \in E \\ 0 & \text{if } (p_i,p_j),(p_j,p_i) \notin E \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Clearly W is a semi-$[\phi,\psi]$-factor of G. Let $W_2 = \{(p_i,p_j)|W(p_i,p_j) = \frac{1}{2}\}$. We will show that if condition (3.8) does not hold, then it is possible to change W, to reduce W_2.

Notice that $d(p_i,W)$ is an integer iff $d(p_i,W_2)$ is even. Thus, if for some j, $d(p_j,W)$ is not integer, then there exists an index k such that p_j and p_k are in the same component of W_2, and $d(p_k,W)$ is not integer. Of course, $a_j + \frac{1}{2} \leq d(p_j,W) \leq b_j - \frac{1}{2}$ and $a_k + \frac{1}{2} \leq d(p_k,W) \leq b_k - \frac{1}{2}$. Hence, by alternating any path in W_2, between p_j and p_k, we reduce W_2.

4. The main theorems

Theorem 4.1: Let $\phi = (a_1,\ldots,a_n)$, $\psi = (b_1,\ldots,b_n)$ be two sequences of non-negative integers, such that $a_i < b_i$ for $i = 1,\ldots,n$. Then a graph G has a $[\phi,\psi]$-factor if and only if for all $S \subseteq N$

$$\sum_{i\in S} a_i \leq \sum_{j=1}^{n} \min(b_j,\delta(p_j,S)). \tag{3.9}$$

Proof: If G has a $[\phi,\psi]$-factor, then condition (3.9) holds, as we showed in the proof of Lemma 3.3. To prove the other direction, suppose condition (3.9) holds and let W be a semi-$[\phi,\psi]$-factor of G, for which condition (3.8) holds. Let $W_2 = \{(p_i,p_j) | W(p_i,p_j) = \frac{1}{2}\}$. Each component of W_2 is Eulerian, and hence has an Eulerian cycle. If $W_2 \neq \emptyset$, let $C = [p_{i_1},\ldots,p_{i_\ell}]$ be such a cycle. If C is even we may reduce W_2 by alternating C, either positively or negatively. If $C = [p_{i_1},\ldots,p_{i_\ell}]$ is odd, then since $a_{i_1} < b_{i_1}$, either $d(p_{i_1},W) > a_{i_1}$ or $d(p_{i_1},W) < b_{i_1}$ (or both). In the first case a negative alternation of C will reduce W_2 and in the second case, a positive alternation will reduce W_2.

Notice that the strong inequalities $a_i < b_i$ are needed for the reduction only for the alternation of odd cycles. Since the smallest odd cycle is a triangle we may allow an equality $a_i = b_i$ in one or two indices i, in the conditions of Theorem 4.1.

Theorem 4.2: Suppose $\phi \neq \psi$ and ϕ is arranged in a non-increased order. Then, a $[\phi,\psi]$-realization exists if and only if

a) for $j = 1,\dots,n$

$$\sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} \min(b_i, j-1) + \sum_{i=j+1}^{n} \min(b_i, j) \tag{4.1}$$

and

b) there are no two disjoint sets $S,T \subset N$, $S \neq \emptyset$, for which

$$\sum_{i \notin T} a_i + \sum_{i \in T} b_i \text{ is odd}, \tag{4.2}$$

$$\sum_{i \in S} a_i = \sum_{i \in T} b_i + s(n-1-t), \tag{4.3}$$

and

$$a_i = b_i \quad \text{for} \quad i \in N-S-T. \tag{4.4}$$

Proof: Condition (4.1) is a particular case of condition (3.9), where the given graph is K_n.

Suppose a $[\phi,\psi]$-realization exists and for $S_0,T_0 \subset N$ such that $S_0 \neq \emptyset$ and $S_0 \cap T_0 = \emptyset$, conditions (4.2)-(4.4) hold. Then

$$s_0(n-1-t_0) = \sum_{i \in S_0} a_i - \sum_{i \in T_0} b_i \leq \sum_{i \in S_0} d(p_i,H) - \sum_{i \in T_0} d(p_i,H) \leq s_0(n-1-t_0).$$

(The last inequality follows from Lemma 3.1.) Thus,

$$d(p_i, H) = a_i \quad \text{for} \quad i \in N-T$$

and

$$d(p_i, H) = b_i \quad \text{for} \quad i \in T.$$

Hence, by condition (4.2), $\sum_{i=1}^{n} d(p_i, H)$ is odd, a contradiction.

Suppose now that condition (4.1) holds for ϕ and ψ, but no $[\phi,\psi]$-realization exists. We will construct a pair of disjoint sets $S,T \subset N$, $S \neq \emptyset$, for which conditions (4.2)-(4.4) hold.

By Lemma 3.3 and the proof of Theorem 4.1, there exists a semi-$[\phi,\psi]$-realization W such that $d(p_i,W)$ is an integer for $i = 1,\ldots,n$, and such that $W_2 = \{(p_i,p_j) | W(p_i,p_j) = \frac{1}{2}\}$ has no even cycles. Suppose $C_1 = [p_{i_1},\ldots,p_{i_\ell}]$ and $C_2 = [p_{j_1},\ldots,p_{j_k}]$ are two odd Eulerian cycles in W_2. If $W(p_{i_1},p_{j_1}) = 1$ we may reduce W_2 by defining $W(p_{i_1},p_{j_1}) = 0$ and making a positive alternation of both C_1 and C_2. (See Figure 1). A similar

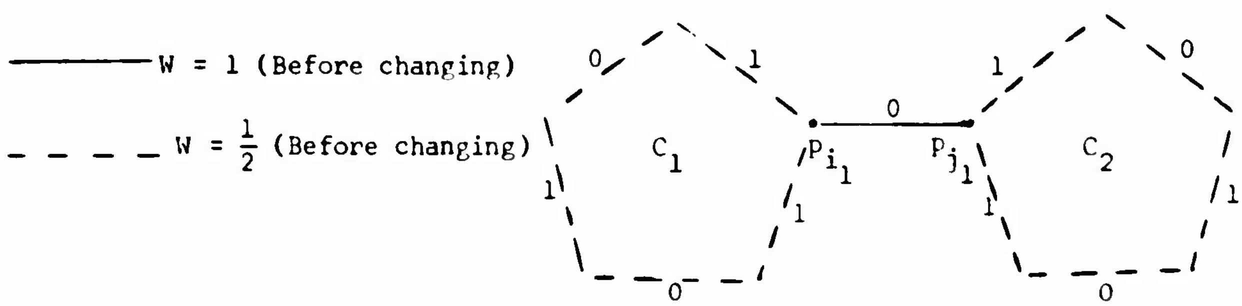

Figure 1

The numbers show the new values of W.

reduction is obtained if $W(p_{i_1},p_{j_1}) = 0$ by setting $W(p_{i_1},p_{j_1}) = 1$ and making a negative alternation of the cycles. Thus there exists a semi-$[\phi,\psi]$-realization W, in which $W_2 = [p_{i_1},\ldots,p_{i_\ell}] = C$ is an odd cycle. Fix W for which W_2 is maximal (and fix C).

Let

$$S = \{i \mid p_i \notin W_2,\ W(p_i,p_{i_1}) = \dots = W(p_i,p_{i_\ell}) = 1\},$$

$$T = \{i \mid p_i \notin W_2,\ W(p_i,p_{i_1}) = \dots = W(p_i,p_{i_\ell}) = 0\}.$$

We have to show that if no $[\phi,\psi]$-realization exists, then conditions (4.2)-(4.4) hold for S and T. We will show it, in a chain of 5 claims, the proofs of which will be given at the end of the proof of the theorem.

Claim 1. If $i \in S$ and $j \in N-T$, then $W(p_i,p_j) = 1$.

Claim 2. If $i \in T$ and $j \in N-S$, then $W(p_i,p_j) = 0$.

Claim 3. If $i \notin S$, then $d(p_i,W) = b_i$.

Claim 4. If $i \notin T$, then $d(p_i,W) = a_i$.

Claim 5. The set S is not empty.

Using Lemma 3.2, claims 1, 2 and 5 imply the equality $\sum_{i\in S} d(p_i,W) = \sum_{i\in T} d(p_i,W) + s(n-1-t)$. Claims 3 and 4 imply that $\sum_{i\in S} a_i = \sum_{i\in S} d(p_i,W)$, $\sum_{i\notin T} a_i = \sum_{i\notin T} d(p_i,W)$, $\sum_{i\in T} b_i = \sum_{i\in T} d(p_i,W)$, and for $i \notin S \cup T$, $a_i = d(p_i,W) = b_i$. Therefore conditions (4.3) and (4.4) hold for S and T and since W has exactly one odd cycle on which $W = \frac{1}{2}$, the sum $\sum_{i=1}^{n} d(p_i,W)$ is odd, and hence condition (4.2) also holds. Thus we have shown that if no $[\phi,\psi]$-realization exists we can find a pair $S,T \subset N$ for which conditions (4.2)-(4.4) hold.

To finish the proof we have only to prove the 5 claims:

<u>Proof of claim 1</u>: Since $i \in S$, $(p_i,p_j) \notin W_2$. Suppose $W(p_i,p_j) = 0$. Then by the construction of S, $p_j \notin W_2$. As $j \notin T$ there exists a vertex p_k on W_2 such that $W(p_j,p_k) = 1$. Let p_r be the consecutive vertex of p_k, along C. (See Figure 2). By redefining $W(p_k,p_r) = 1$, $W(p_i,p_j) = W(p_j,p_k) = W(p_i,p_r) = \frac{1}{2}$ we enlarge W_2, a contradiction to its maximality.

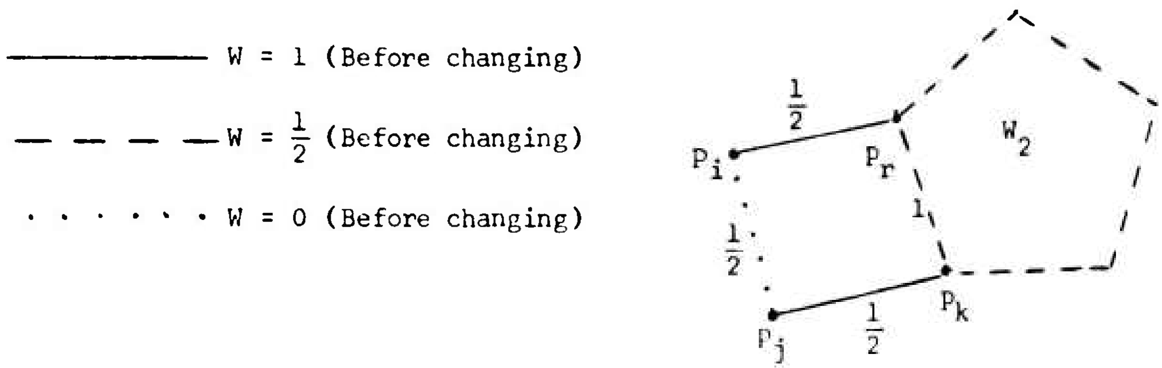

Figure 2

The numbers show the new values of W.

<u>Proof of claim 2</u>: Similarly to the case of claim 1, if $W(p_i,p_j) \neq 0$ for $i \in T$, $j \in N-S$, then $W(p_i,p_j) = 1$, $p_j \notin C$, there exists k such that $W(p_j,p_k) = 0$ and if p_r is the consecutive vertex of p_k, on C, then redefining $W(p_k,p_r) = 0$ and $W(p_i,p_j) = W(p_j,p_k) = W(p_i,p_r) = \frac{1}{2}$ enlarges W_2. (See Figure 3).

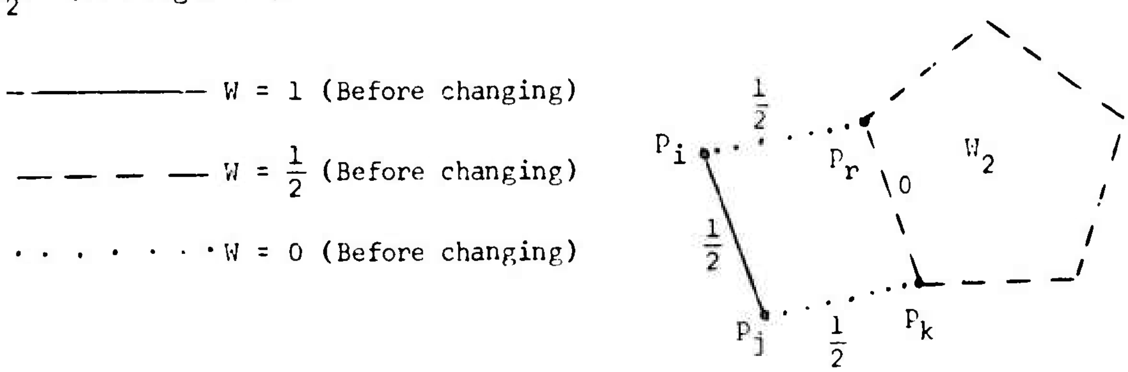

Figure 3

The numbers show the new values of W.

Proof of claim 3: Suppose $i \notin S$ and $d(p_i,W) < b_i$. If $p_i \in C$ we may reduce W_2 by a positive alternation of C, beginning at p_i. (Compare to the proof of Theorem 4.1). If $p_i \notin C$ then $W(p_i,p_k) = 0$ for some $p_k \in C$ and we may once more reduce W_2, by a negative alternation of C, beginning at p_k and by set $W(p_i,p_k) = 1$. (See Figure 4). In each case we obtain a semi-graph W' which never gets the value $\frac{1}{2}$, i.e. a $[\phi,\psi]$-factor, a contradiction.

— — — — $W = \frac{1}{2}$ (Before changing)

. . . . $W = 0$ (Before changing)

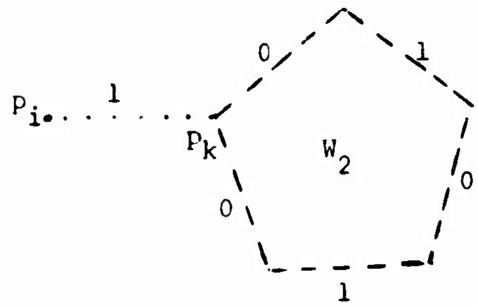

Figure 4

The numbers show the new values of W.

Proof of claim 4: As in the case of claim 3, if $d(p_i,W) > a_i$, $p_i \notin C$ but there exists an index k such that $p_k \in C$ and $W(p_i,p_k) = 1$. Thus we may reduce W_2 by letting $W(p_i,p_k) = 0$ and by making a negative alternation of C, beginning at p_k.

Proof of claim 5: We assumed $\phi \neq \psi$. Let $a_k \neq b_k$. Claims 3 and 4 imply that $k \in S \cup T$. If $k \in S$, then there is nothing to prove. Suppose $k \in T$. Then $d(p_k,W) = b_k > a_k \geq 0$, and since $W(p_k,p_j) = 0$ for all $j \in N-S$, there exists $r \in S$ such that $W(p_k,p_r) = 1$. In particular S is not empty.

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